HISTORICAL PROJECTS IN DISCRETE MATHEMATICS

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ABSTRACT

Presented is a summary of two historical curricular modules for undergraduate discrete mathematics. The first "Deduction through the Ages" is a discussion of how modern mathematics arrived at the truth of an implication (an "if-then" statement) in propositional logic. The second "Networks and Spanning Trees" presents motivational material for the definition, enumeration, and application of trees in graph theory.

Keywords: Propositional logic, implication, graph theory, labeled trees, minimal spanning tree, Borůvka's algorithm.

1 Introduction

In this talk we discuss teaching discrete mathematics from primary historical sources and provide the results of a statistical study concerning the impact of this pedagogical technique on student learning attitudes. Over the past four years our interdisciplinary, intercollegiate team of seven faculty have developed 18 curricular modules that incorporate passages from primary sources to teach core content in finite mathematics, combinatorics, logic, abstract algebra, algorithm design, and computer science courses. This builds on a pilot study to teach from historical projects [3]. Each module is designed around one or several historical sources and develops a key concept (or several concepts) in the curriculum by examining the work of the pioneers and offering student exercises that illuminate and extrapolate from the source. Topics for the modules are often an examination of the ideas behind modern definitions, algorithms or lemmas that appear as opaque or unmotivated statements in today's textbooks, such as the truth table of an implication in propositional logic, the definition of tree in graph theory, or the formula for the summation of squares, $\sum_{i=1}^{n} i^2 = (n^3/3) + (n^2/2) + (n/6)$, and the unenlightening proof of this equality by formal mathematical induction. For the complete list of our curricular projects, along with the text of each one, see our web resource [2].

Why teach from historical sources when textbooks offer a concise, mathematically precise presentation of the subject? First, historical sources add context, with the original author keenly motivated to solve a particular problem or find a robust setting for previously fragmented solutions. We read what the problem was and witness a pioneering, often paradigm-setting approach. The primary source reveals the motivation for study of the subject or paradigm. Historical sources add direction to the subject matter. We observe where the author begins, how a problem is solved, and what subsequent work builds on the solution. Additionally we as readers are forced to grapple with the verbal meaning of a passage, consider non-standard formulations of ideas, and ask "What is an appropriate system of notation for this problem?" "What are the key properties to a solution to this problem?" We learn through cognitive dissonance. The thought process required to bridge the gap between the historical and the modern offers an invaluable learning experience. We gain insight into the process of discovery as well as an appreciation of the cultural and intellectual setting in which the author was writing. For further reasons to study from primary historical sources, see [1, 3]. For the results of a pilot study using this pedagogical technique, see [3]. To illustrate how the historical approach can be used to teach mathematical content, we examine two historical modules in detail: "Deduction through the Ages," and "Networks and Spanning Trees." The first is a study of the original work of several philosophers, logicians and mathematicians who have contributed to an understanding of the truth table of an implication (an "if-then" statement). The second examines the notion of tree and its applications before graph theory was an independent subject of study.

2 Deduction through the Ages

While today the truth of $p \to q$ (p implies q) is a matter of settled logic, the ancient Greeks debated at length when the following hypothetical proposition holds: "If a warrior is born at the rising of the Dog Star, then that warrior will not die at sea." The Greek philosopher Philo of Megara (ca. 4th century B.C.E.) maintained that a valid hypothetical proposition is "that which does not begin with a truth and end with a falsehood" [18, II. 110]. The on-line written project "Deduction through the Ages" [2] outlines five argument forms stated by Chrysippus (ca. 280–206 B.C.E.) [11, p. 189], and raises the question (for students and instructors) whether these five rules could be special cases of just one rule. This presentation focuses on the following three (of five) rules:

1. If the first, [then] the second. The first. Therefore, the second.

3. Not both the first and the second. The first. Therefore, not the second.

5. Either the first or the second. Not the first. Therefore, the second.

Verbal argument asserting the equivalence of these rules is difficult, and a more streamlined method for discussing their relation to each other is sought. An old point of view on logic is to reduce the subject to a system of calculation, whereby the rules of reasoning could be automated. The German philosopher, mathematician, and universalist Gottfried Wilhelm Leibniz (1646–1716) was one of the first to pursue this idea, and sought a *characteristica generalis* (general characteristic) or a *lingua generalis* (general language) that would serve as a universal symbolic language and reduce all debate to calculation. This in part served as motivation for Leibniz to introduce his symbols for differentiation and integration.

2.1 Boole's Algebra of Statements

In the modern era, an initial attempt at a symbolic and almost calculational form of elementary logic was introduced by the English mathematician George Boole (1815–1864). Author of *An Investigation of the Laws of Thought* [4, 5], Boole believed that he had reduced language and reasoning to a system of calculation involving the signs "$\times$", "$+$", "$-$", where "$\times$" denotes "and," "$+$" denotes "or," and "$-$" denotes "not." Boole writes [5]:

PROPOSITION I.

All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz:

1st. Literal symbols, as x, y &c., representing things as subjects of our conceptions.

2nd. Signs of operation, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the elements.

3rd. The sign of identity, $=$.

And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra. ...

If x represent any class of objects, then will $1 - x$ represent the contrary or supplementary class of objects, i.e. the class including all objects which are not comprehended in the class x.

The symbols "$\times$", "$+$", "$-$", however, lose their arithmetic meaning when applied to the logic of statements. For example, letting a denote the class of apples and b the class of red objects, then in Boole's notation the class of objects that are not red apples would be $1 - ab$. Objects that are either not apples or not red would be $(1 - a) + (1 - b)$. Thus, in Boole's notation

$$1 - ab = (1 - a) + (1 - b),$$

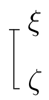
which reflects a statement in logic, not arithmetic. Also, Boole does not introduce a symbol for an "if-then" statement, so writing Chrysippus's first rule in this arithmetic notation is difficult.

2.2 Gottlob Frege Invents a Concept-Script

Let's now turn to the work of the German mathematician and philosopher Gottlob Frege (1848–1925) who sought a logical basis, not for language as Boole, but for mathematics. In *The Basic Laws of Arithmetic* [12], Frege introduces his own system of notation, called a *concept-script* or "Begriffsschrift"in the original German, which shows no kinship with the arithmetical symbols "$\times$", "$+$", "$-$". The centerpiece of Frege's notation is the condition stroke[1]. From *The Basic Laws of Arithmetic*, we read:

§12. **Condition-stroke, And, Neither-nor, Subcomponents, Main Component.**

In order to enable us to designate the subordination of a concept under a concept, and other important relations, I introduce the function of two arguments



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by stipulating that its value shall be the False if the True be taken as ζ-argument and any object other than the True be taken as ξ-argument, and that in all other cases, the value of the function shall be the True. . . . The vertical stroke I call the *condition-stroke*. . . . .

∞∞∞∞∞∞∞∞

Thus, the symbol $\begin{array}{l}\ulcorner B\\ \llcorner A\end{array}$ is false only when A (the beginning proposition) is true and B (the ending proposition) is false. The reader is asked to compare the truth of Frege's condition stroke to Philo's verbal statement that a valid hypothetical proposition is "that which does not begin with a truth and end with a falsehood." The condition stroke is true when it is **not** the case that it begins (A) with a true statement and ends with a false statement (B). Thus, the condition stroke is Frege's symbol for an implication (a hypothetical proposition in ancient Greece). We use a few other symbols from the "Begriffsschrift." A horizontal line — denotes a "judgment stroke" that renders the value of either true or false when applied to a proposition. For example, $— 2^2 = 5$ returns the value "false," while $— 2^2 = 4$ returns "true." The symbol $\top\,\xi$ denotes the negation of $—\,\xi$, while $\vdash \zeta$ denotes that ζ is a true statement. These symbols may be combined in what Frege calls "amalgamation of horizontals," so that $\vdash (\top\,\Delta)$ becomes $\vdash\!\top\,\Delta$, meaning that the negation of Δ is true, i.e., Δ itself is false.

Let's now write Chrysippian rules (1), (3), and (5) above entirely in the concept-script. Frege himself states the "First Method of Inference" as from the propositions $\begin{array}{l}\ulcorner B\\ \llcorner A\end{array}$ and $\vdash A$ we may infer $\vdash B$. Letting A denote "the first" and B denote "the second," this "First Method of Inference" becomes verbally: "If the first, then the second. The first is true, therefore, the second is true." How can we write Chrysippus's third rule in Frege notation? Recall that the symbol

$$\begin{array}{l}\ulcorner B\\ \llcorner A\end{array}$$

is false only when A is true and B is false. Thus $\begin{array}{l}\ulcorner\!\top B\\ \llcorner A\end{array}$ is false only when A is true and B is true, which has the same truth value as "not both A and B." Again, letting A denote "the first" and B denote "the second," we see that "not both the first and the second, not the first, therefore, not the second" can be written as from $\begin{array}{l}\ulcorner\!\top B\\ \llcorner A\end{array}$ and $\vdash A$, it follows $\vdash\!\top B$. Finally, to write the fifth Chrysippian rule in the concept-script, note that the symbol $\begin{array}{l}\ulcorner B\\ \llcorner\!\top A\end{array}$ is false only when A is false and B is false, which has the same truth value as "either the first or the second," using the inclusive "or." Thus, "either the first or the second, not the first, therefore, the second" can be rendered as from $\begin{array}{l}\ulcorner B\\ \llcorner\!\top A\end{array}$ and $\top\,A$, it follows $\vdash B$.

Thus, rules (1) , (3) and (5) can all be written using the same root symbol, the condition stroke, and minor variations on negating or asserting its arguments. This demonstrates the interconnectedness of these rules, and offers insight into their possible equivalence.

2.3 Russell and Whitehead Find New Notation

While somewhat awkward in execution, Frege's condition stroke advances his philosophy that mathematical truths should follow from truths in logic, a point of view known today as logicism. Two later practitioners of logicism whose work set the stage for mathematical logic of the twentieth century were Bertrand Russell (1872–1970) and Alfred North Whitehead (1861–1947). Russell was a prolific writer, contributing to the fields of education, history, religion, and political theory, not to mention philosophy and logic. Let's read a short excerpt from Russell and Whitehead's monumental collaboration *Principia Mathematica* [17], where an implication (an "if-then" statement) is formally defined. Note how the definition of "p implies q" reduces to the equivalent inclusive "or" statement in Frege's notation.

The fundamental functions of propositions. . . .

[T]here are four special cases which are of fundamental importance, since all the aggregations of subordinate propositions into one complex proposition which occur in the sequel are formed out of them step by step.

They are (1) The Contradictory Function, (2) the Logical Sum or Disjunctive Function, (3) the Logical Product, or Conjunctive Function, (4) the Implicative Function. . . .

The Contradictory Function with argument p, where p is any proposition, is the proposition which is the contradictory of p, that is, the proposition asserting that p is not true. This is denoted by $\sim p$. Thus, $\sim p$ . . . means the negation of the proposition p. It will also be referred to as the proposition not-p. . . .

The Logical Sum is a proposition with two arguments p and q, and is the proposition asserting p or q disjunctively, that is, asserting that at least one of the two p and q is true. This is denoted $p \vee q$. . . . Accordingly $p \vee q$ means that at least p or q is true, not excluding the case in which both are true.

The Logical Product is a propositional function with two arguments p and q, and is the proposition asserting p and q conjunctively, that is, asserting that both p and q are true. This is denoted by $p.q$ . . . . Accordingly $p.q$ means that both p and q are true. . . .

The Implicative Function is a propositional function with two arguments p and q, and is the proposition that either not-p or q is true, that is, it is the proposition $\sim p \vee q$. Thus, if p is true, $\sim p$ is false, and accordingly the only alternative left by the proposition $\sim p \vee q$ is that q is true. In other words if p and $\sim p \vee q$ are both true, then q is true. In this sense the proposition $\sim p \vee q$ will be quoted as stating that p implies q. The idea contained in this propositional function is so important that it requires a symbolism which with direct simplicity represents the proposition . . . . The symbol employed for "p implies q", i.e. for "$\sim p \vee q$" is "$p \supset q$." This symbol may also be read "if p, then q." . . .

But this . . . by no means determines whether anything, and if so what, is implied by a false proposition. What it does determine is that if p implies q, then it cannot be the case that p is true and q is false, ...

With these crisp definitions, Chrysippus's rules can be written as follows in the notation of *Principia Mathematica*:

$$1.\ \ p \supset q,\ p,\quad \therefore q$$
$$3.\ \ \sim(p.q),\ p,\quad \therefore \sim q$$
$$5.\ \ p \vee q,\ \sim p,\quad \therefore q.$$

To discuss the relation between rules (1) and (5), note that from [17] every implication is equivalent to a certain inclusive "or" statement and vice versa.

$$p \supset q \equiv \sim p \vee q, \qquad p \vee q \equiv \sim p \supset q.$$

The relation between rules (1) and (3) can be discovered from the equivalence between an implication and a negated "and" statement and vice versa.

$$p \supset q \equiv \sim p \vee q \equiv \sim (p.(\sim q)), \qquad \sim(p.q) \equiv \sim p \vee \sim q \equiv p \supset \sim q.$$

Thus, the major premise of rule (1), "if the first, then the second" is equivalent to a certain inclusive "or" statement, which in turn is equivalent to a certain negated "and" statement. Of course, the individual arguments of these "or" and "and" statements may themselves be negated, as we saw when discussing writing Chrysippus's rules in the "Begriffsscrift."

2.4 Post Develops Truth Tables

Emil Post (1897–1954) developed a highly efficient method to represent the truth values of compound statements involving the connectives "and," "or," "not," and "if-then." He dubbed these schematic representations "truth tables," a term which is in current use today. Emil was born in Poland, of Jewish parents, with whom he emigrated to New York in 1904. He received his doctorate from Columbia University, where he participated in a seminar devoted to the study of *Principia Mathematica*. In his dissertation of 1921, "Introduction to a General Theory of Propositional Functions" [14], he develops the notion of truth tables and clearly displays the table for an implication. With this in hand, the equivalence of the major premises in Chrysippus's rules is reduced to mere calculation of truth values.

INTRODUCTION TO A GENERAL THEORY OF ELEMENTARY PROPOSITIONS.

BY EMIL L. POST.

INTRODUCTION.

In the general theory of logic built up by Whitehead and Russell [17] to furnish a basis for all mathematics there is a certain subtheory . . . this subtheory uses . . . but one kind of entity which the authors have chosen to call elementary propositions. . . .

2. Truth-Table Development—Let us denote the truth-value of any proposition p by $+$ if it is true and by $-$ if it is false. This meaning of $+$ and $-$ is convenient to bear in mind as a guide to thought, ... Then if we attach these two primitive truth-tables to $\sim$ and $\vee$ we have a means of calculating the truth-values of $\sim p$ and $p \vee q$ from those of their arguments.

p	$\sim p$
$+$	$-$
$-$	$+$

$p,\ q$	$p \vee q$
$+\ +$	$+$
$+\ -$	$+$
$-\ +$	$+$
$-\ -$	$-$

... It will simplify the exposition to introduce ...

$$p \supset q \,.=.\, \sim p \vee q$$

read "p implies q," ... having the table

$p,\ q$	$p \supset q$
$+\ +$	$+$
$+\ -$	$-$
$-\ +$	$+$
$-\ -$	$+$

With the truth table of an implication we have arrived, after more than two millennia of deductive thought, where modern discrete mathematics textbooks begin a discussion of propositional logic. It is now a textbook exercise to verify, via truth tables, that the following logical equivalence holds:

$$p \supset q \equiv \sim p \vee q, \qquad p \vee q \equiv \sim p \supset q$$
$$p \supset q \equiv \sim (p.(\sim q)), \qquad \sim (p.q) \equiv p \supset \sim q.$$

3 Networks and Spanning Trees

In 1857 Arthur Cayley (1821–1895) published a paper [9] that introduces the term "tree" to describe the logical branching that occurs when iterating the fundamental process of (partial) differentiation. Of composing four symbols that involve derivatives, Cayley writes "But without a more convenient notation, it would be difficult to find [their] corresponding expressions . . . . This, however, can be at once effected by means of the analytical forms called trees . . . " [9]. Without defining the term "tree," Cayley has identified a certain structure that occurs today in quite different situations, from networks in computer science to representing efficient delivery routes for transportation. In a later paper "A Theorem on Trees" [10] published in 1889, Cayley counts trees in which every node (vertex) carries a fixed name or label, arriving at a result that today is known as "Cayley's formula" for the number of labeled trees on n vertices. His proof is a bit incomplete, and we discuss the work of Heinz Prüfer (1896–1934) on counting labeled trees via an enumeration of certain railway networks [15]. This is followed by a discussion of Otakar Borůvka's (1889–1995) work on finding a net of least total edge length, i.e., a minimal spanning tree, from all labeled trees on n fixed vertices [6].

3.1 Prüfer's Enumeration of Trees

The German mathematician Heinz Prüfer offers a quite clever and geometrically appealing method for counting what today are called labeled trees. He uses no modern terminology, not even the word "tree" in his work. Instead, the problem is introduced via an application [16]: Given a country with n-many towns, in how many ways can a railway network be constructed so that

1. the least number of railway segments is used; and
2. a person can travel from each town to any other town by some sequence of connected segments.

The ideas expressed here, that the least number of railway segments is used, yet travel remains possible between any two towns, are recognized today as properties that characterize such a railway network as a tree. Since the towns are fixed, their names (labels) are not interchangeable, and a labeled tree is an excellent model for this problem. Prüfer wishes to count all railway networks satisfying properties (1) and (2) above, and in doing so, he arrives at a result that agrees with Cayley's formula. Prüfer assigns to each tree a particular symbol based on the point labels (town names). Counting the resulting symbols is then much easier than counting trees. Of course, establishing a one-to-one correspondence between symbols and trees requires some work, which Prüfer writes "follows from an induction argument" (on the number of towns). Let's read a brief excerpt from "A New Proof of a Theorem about Permutations" [15, 16]:

[We] assign to each railway network, in a unique way, a symbol $\{a_1,\ a_2,\ \ldots\ ,\ a_{n-2}\}$, whose $n-2$ elements can be selected independently from any of the numbers 1, 2, ..., n. There are n^{n-2} such symbols, and this fact, together with the one-to-one correspondence between networks and symbols, will complete the proof.

In the case $n = 2$, the empty symbol corresponds to the only possible network, consisting of just one single segment that connects both towns. If $n > 2$, we denote the towns by the numbers 1, 2, ..., n and specify them in a fixed sequence. The towns at which only one segment terminates we call the endpoints. ...

In order to define the symbol belonging to a given net for $n > 2$, we proceed as follows.

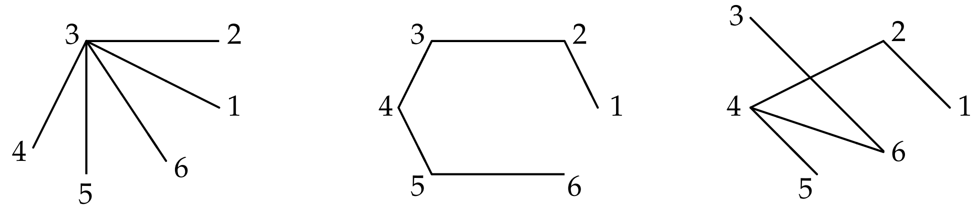
Let b_1 be the first town which is an endpoint of the net, and a_1 the town which is directly joined to b_1. Then a_1 is the first element of the symbol. We now strike out the town b_1 and the segment $b_1\,a_1$. There remains a net containing $n-2$ segments that connects $n-1$ towns in such a way that one can travel from each town to any other.

If $n-1 > 2$ also, then one determines the town a_2 with which the first endpoint b_2 of the new net is directly connected. We take a_2 as the next element of the symbol. Then we strike out the town b_2 and the segment $b_2\,a_2$. We obtain a net with $n-3$ segments and the same properties.

We continue this procedure until we finally obtain a net with only one segment joining 2 towns. Then nothing more is included in the symbol.

Examples:

Nets:



Symbols: $\{3,\ 3,\ 3,\ 3\}$ $\{2,\ 3,\ 4,\ 5\}$ $\{2,\ 4,\ 6,\ 4\}$

3.2 Borůvka's Solution to a Minimization Problem

In 1926 Otakar Borůvka (1899–1995) published [6, 7] the solution to an applied problem of immediate benefit for constructing an electrical power network in the Southern Moravia Region, now part of the Czech Republic. In recalling his own work, Borůvka writes [8, 13]:

> My studies at polytechnical schools made me feel very close to engineering sciences and made me fully appreciate technical and other applications of mathematics. Soon after the end of World War I, at the beginnings of the 1920s, the Electrical Power Company of Western Moravia, Brno, was engaged in rural electrification of Southern Moravia. In the framework of my friendly relations with some of their employees, I was asked to solve, from a mathematical standpoint, the question of the most economical construction of an electric power network. I succeeded in finding a construction . . . which I published in 1926 . . . .

Let's examine specifically how Borůvka phrased the problem [7]:

> There are n points in the plane (in space) whose mutual distances are all different. We wish to join them by a net such that:
>
> 1. Any two points are joined either directly or by means of some other points.
> 2. The total length of the net would be the shortest possible.

Thus, of all n^{n-2} labeled trees on n points (towns), which tree(s) has (have) the shortest possible total edge length. Borůvka proposes a simple algorithm to find such a net of minimal total length, based on the guiding principle "I shall join each of the given points with the point nearest to it" [7].

A Contribution to the Solution of a Problem on the Economical Construction of Power Networks

Dr. Otakar Borůvka

In my paper "On a Certain Minimal Problem," I proved a general theorem, which, as a special case solves the following problem:

There are n points in the plane (in space) whose mutual distances are all different. We wish to join them by a net such that:

1. *Any two points are joined either directly or by means of some other points.*
2. *The total length of the net would be the shortest possible.*

It is evident that a solution of this problem could have some importance in electrical power network designs; hence I present the solution briefly using an example. . . . .

I shall give the solution of the problem in the case of 40 points[2] given in Fig. 1.

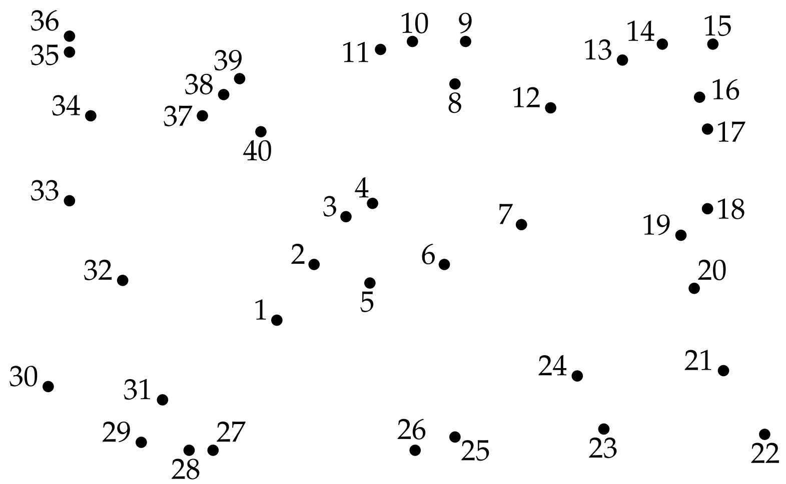


Fig. 1.

I shall join each of the given points with the point nearest to it. Thus, for example, point 1 with point 2, point 2 with point 3, point 3 with point 4 (point 4 with point 3), point 5 with point 2, point 6 with point 5, point 7 with point 6, point 8 with point 9 (point 9 with point 8), etc. I shall obtain a sequence of polygonal strokes 1, 2, . . . , 13 (Fig. 2).

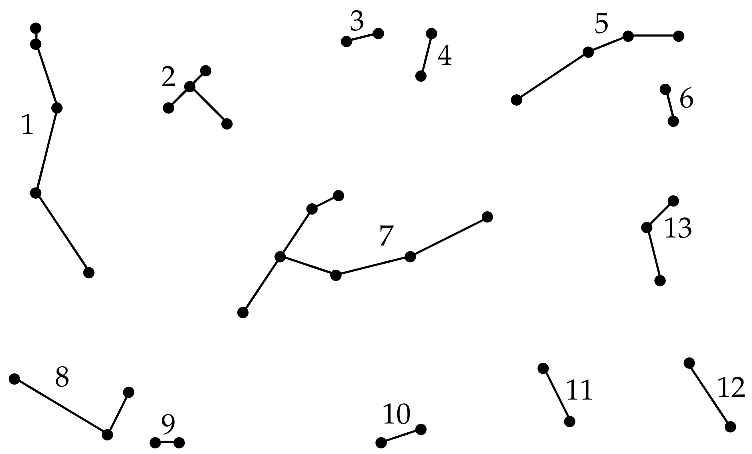


Fig. 2.

I shall join each of these strokes with the nearest stroke in the shortest possible way. Thus, for example, stroke 1 with stroke 2 (stroke 2 with stroke 1), stroke 3 with stroke 4 (stroke 4 with stroke 3), etc. I shall obtain a sequence of polygonal strokes 1, 2, 3, 4 (Fig.3).

I shall join each of these strokes in the shortest way with the nearest stroke. Thus stroke 1 with stoke 3, stroke 2 with stroke 3 (stroke 3 with stroke 1), stroke 4 with stroke 1. I shall finally obtain a single

[2]Borůvka only labeled the points 1 through 9 in his original paper. We have included labels of all points for later reference.

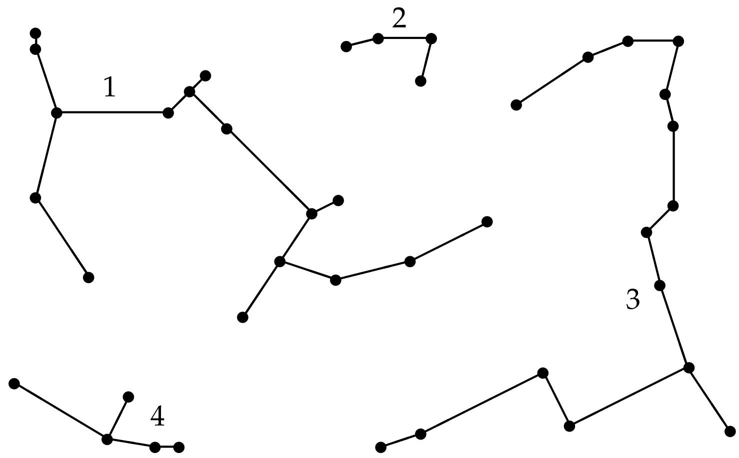


Fig. 3.

polygonal stroke (Fig. 4)[3] which solves the given problem.

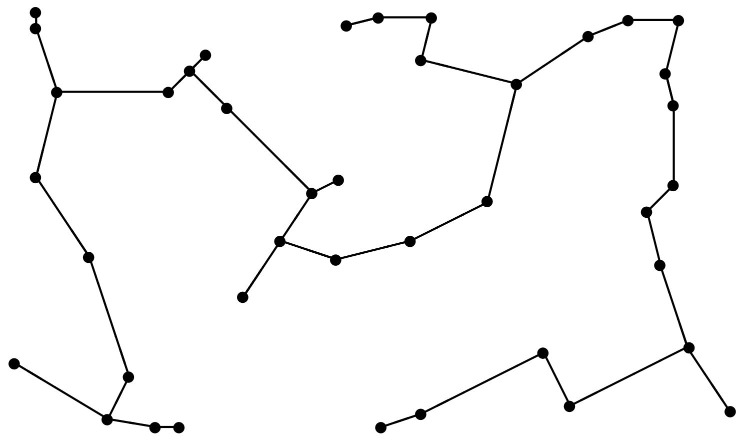


Fig. 4.

4 Impact on Student Leaning Attitudes

Over the past four years the impact of our historical projects on student learning and attitudes has been assessed by our statistical consultant, Dr. David Trafimow of the Department of Psychology, New Mexico State University. Students are asked to complete a pre- and post-course questionnaire which are matched by the use of anonymous codes. A sample question includes:

[3]In the original paper [7], Figure 4 is rotated 180°.

Which best describes you:
I am capable (Extremely) : (Quite) : (Slightly) : (Neutral) : (Slightly) : (Quite) : (Extremely) incapable of explaining Math/Computer Science concepts in writing.

Above, "extremely capable" is given the value of 3, "quite capable" the value of 2, . . . , and "extremely incapable" the value -3, forming a scale from $+3$ to -3, with 0 being neutral. First the questionnaire was shown to be reliable by repeating similar questions, and Cronbach's alpha reliability factor was .89 on the pre- and .90 on the post-course questionnaires, where an alpha factor greater than .7 is considered reliable. On the scale from $+3$ to -3 above, the mean from pre- to post-course questionnaire increased from 1.13 to 1.47. Given the null hypothesis that there is no difference from pre- to post-course questionnaires, the paired T-test between the means of these two questionnaires yields $p < .001$, indicating that the probability of the difference occurring by chance is less than 1 in 1000. Our consultant reaches the conclusion that students' estimates of their Math/Computer Science understanding increased from pre- to post-test for courses using historical projects.

Acknowledgment

The development of curricular materials for discrete mathematics has been partially supported by the National Science Foundation's Course, Curriculum and Laboratory Improvement Program under grants DUE-0717752 and DUE-0715392 for which the authors are most appreciative. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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