TEACHING NUMBER THEORY FROM SOPHIE GERMAIN'S MANUSCRIPTS: a guided discovery pedagogy

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ABSTRACT

I will discuss a number theory course taught primarily through student study of Sophie Germain's research manuscripts and letters on Fermat's Last Theorem. Students learned everything in a first number theory course from original sources. I will discuss how original sources enhanced guided discovery and just-as-needed pedagogy.

Keywords: Number theory, Sophie Germain, Fermat's Last Theorem, manuscripts, primary sources, original sources, just-in-time, guided discovery, non-lecture, pedagogy

Figure 1. Sophie Germain (1776–1831)

1   Introduction

I will describe the pedagogy of a number theory course taught entirely through studying original sources, primarily Sophie Germain's early nineteenth century research manuscripts and letters on

Fermat's Last Theorem. The course was taught as a mystery, with students as the detectives. A more detailed paper, with greater emphasis on the mathematical content of the manuscripts and the course, is in progress [10].

In all my courses, I aim to have my students study their mathematics directly from primary historical sources, and I have also moved away from lecturing to a classroom active with "guided inquiry" and "just-in-time" discovery. My goal is to dispense with textbooks presenting a purely modernized treatment, which often focuses on answers to questions not asked. I wish to base my courses on student discovery through primary sources aimed at answering meaningful questions [1, 2, 12]. Here I analyze how a method of "guided discovery" to learn mathematics "as needed" interacted with studying Germain's original manscripts in number theory, and discuss the student response.

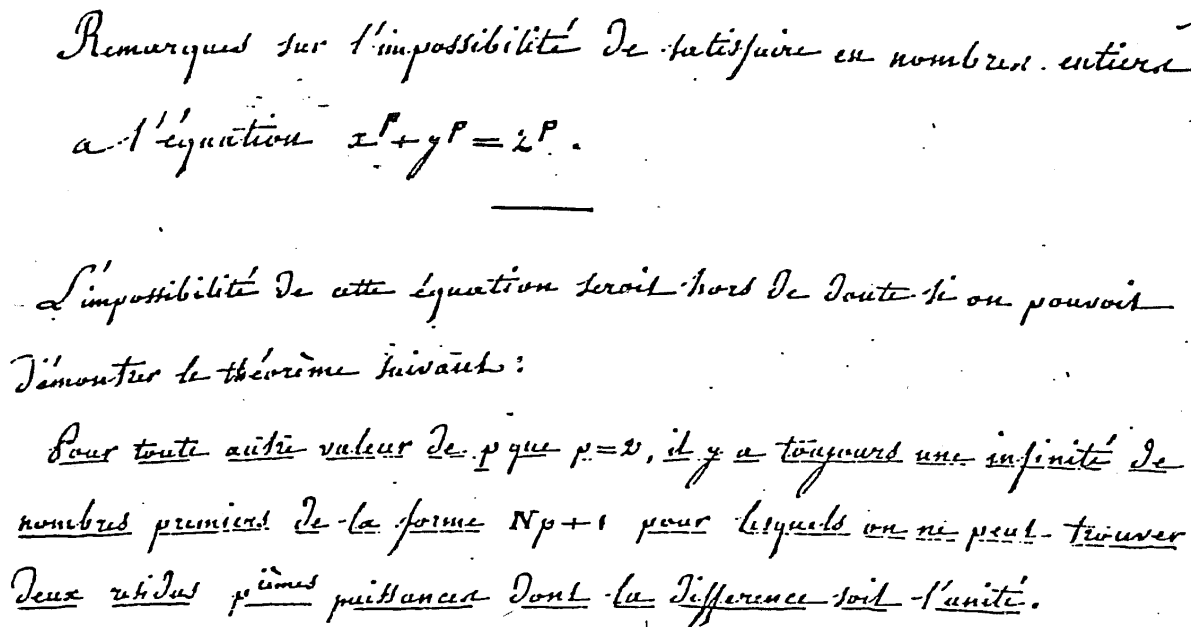## 2   A number theory course à la Sophie Germain

Figure 2. Beginning of Sophie Germain's Manuscript A

Sophie Germain (1776–1831) was the first woman to do important original mathematical research [3]. In number theory she has been known only from a single unpublished result (today called "Germain's Theorem") toward proving Fermat's Last Theorem[1]. Recently, Germain's unpublished manuscripts and letters have revealed that she pursued an ambitious "grand plan" to prove Fermat's Last Theorem [3, 4, 6, 7]. I wondered if one could try teaching number theory largely with Germain's manuscripts.

Recently I was able to teach the standard beginning one semester number theory course at New Mexico State University. The course is at the advanced undergraduate and beginning graduate level. But as a first number theory course, the prerequisite is only a little abstract algebra, and I relied only on student facility at proving theorems.

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[1]In the seventeenth century Fermat claimed that for a natural number $n > 2$, there are no natural number solutions to $x^n + y^n = z^n$. He was finally proven right by Andrew Wiles at the end of the twentieth century [6].

Let us compare the topics in a first course with Sophie Germain's research manuscripts. Germain was writing for experts, and was one of the first to utilize Gauss's congruence view. To understand her manuscripts and letters on Fermat's Last Theorem requires knowing unique factorization, Pythagorean triples, modular arithmetic, Fermat's Little Theorem, Lagrange's Theorem on the number of modular polynomial roots, modular roots of unity, and primitive roots modulo a prime. These are the topics of a first number theory course, along with the Quadratic Reciprocity Law. Conversely, if one understands the topics in a modern first course, one is equipped to understand Germain. Her research foundation is essentially what constitutes a first course today.

Germain was writing for readers such as Gauss and Legendre, so she did not develop any of the above topics in her manuscripts. Rather she assumed that her reader was already familiar with them, and used them freely in her writings. Thus it is more accurate to say that I taught "to" Germain's writings, not "from" them.

I told my students that their challenge was largely to understand Germain's progress towards proving Fermat's Last Theorem. This would be a detective story, because between any two Germain sentences there might be weeks of students learning what they needed in order to make the next leap. I planned to guide my students to learn all the topics in a first course by struggling to understand Germain's writings, with ancillary primary sources providing supplementary material on the Quadratic Reciprocity Law [5].

Next I provide just a small sample of Germain's writings, to give a sense of what students were challenged with, and what her grand plan was for proving Fermat's Last Theorem. A much more detailed picture of the source material and how it created the course content is provided in [10].

Already in the first few lines of Manuscript A of [7] (Figure 2) Germain gives away the big picture of her overall plan for proving Fermat's Last Theorem.

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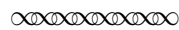
Sophie Germain, from Manuscript A

Remarks on the impossibility of satisfying in whole numbers the equation $x^p + y^p = z^p$.

The impossibility of this equation would follow without doubt if one could demonstrate the following theorem:

For every value of p other than $p = 2$, there is always an infinity of prime numbers of the form $Np + 1$ for which one cannot find two p-th power residues whose difference is unity.

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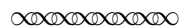
Just this much provokes the first several weeks of course study, including unique factorization, Pythagorean triples, residues, power residues, and inverses modulo primes. Students spend a lot of time deciphering and proving this claim.

In a nutshell, Germain's reason for her claim is what I call her

Basic Lemma. *Suppose $x^p + y^p = z^p$, and that θ is a prime satisfying*

Condition N-C. *There do **not** exist two nonzero **consecutive** p^{th} power residues, upon division by the prime θ.*

Then one of x, y, z is divisible by θ.

From this lemma, which students can prove, it is clear that if there are infinitely many primes $\theta = Np + 1$ satisfying Condition N-C, then since each of these would divide one of x, y, z, no such x, y, z can exist. Germain believed that infinitely many such primes θ existed, thus proving Fermat's Last Theorem. This was her grand plan, and most of her work was devoted to carrying it out.

Germain's extensive manuscripts, and her letters to Gauss (Figures 3, 4) and Legendre [7], lead to a multitude of student questions and investigations. These require Gauss' congruence viewpoint, and the existence of primitive roots for a prime modulus, which Germain uses heavily in her detailed analysis.

Figure 3. Germain's 1819 letter to Gauss

Students also study Germain's proof in her manuscripts of the unpublished "Germain's Theorem" on which her reputation formerly rested. She proves what we today call Case 1 of Fermat's Last Theorem under certain hypotheses, and also something more powerful, a "large size" theorem, "the necessity that the same numbers x, y, and z would be extremely large numbers". However, there is a

Voici ce que j'ai trouvé:

1° L'ordre dans lequel les résidus (puissances égales à l'exposant) se trouvent placés dans la série des nombres naturels Déterminer les diviseurs nécessaires qui appartiennent aux nombres entre lesquels on établit non seulement l'équation de fermat mais encore beaucoup d'autres équations analogues à celle là.

Prenons pour exemple l'équation même de fermat qui est la plus simple de toutes celles dont il s'agit ici. Soit donc, p étant un nombre premier, $z^p = x^p + y^p$.

Je dis que si cette équation est possible, tout nombre premier de la forme $2Np+1$ (N étant un entier quelconque) pour lequel il n'y aura pas deux résidus $p^{ième}$ puissance placés de suite dans la série des nombres naturels divisera nécessairement l'un des nombres x y et z.

Cela est évident, car l'équation donne $z^p = x^p + y^p$ donne la congruence $1 \equiv r^{sp} - r^{tp}$ dans laquelle r représente une racine primitive et s et t des entiers.

Figure 4. "Here is what I have found" from Germain's letter to Gauss

flaw in Germain's proof of the large size of solutions, and one of the challenges to students is to find and understand this.

In sum, understanding Germain's manuscripts ultimately entails almost all the topics of a first number theory course.

3   A pedagogy of just-in-time guided discovery

I integrated student study of Germain's manuscripts with non-lecture pedagogies [11]. The following pedagogies meld, and support each other. However, I will discuss them individually, including how they integrated with Germain's primary sources.

- *A question and inquiry based curriculum, with discovery guidance:*

  I wanted a curriculum driven by student investigation of meaningful questions, not by an instructor or book providing answers to unasked questions. Germain's manuscripts were perfect for this, with the larger question of Fermat's Last Theorem motivating understanding Germain's work, leading to numerous questions as she pursues her goal. Students are faced with numerous questions about how Germain knows the things she claims. I endeavored not to prove anything for students, but rather for them to learn everything through their own discoveries, with just the right tasks and guidance from me. I tried always to keep students in the driver's seat, with me charting the path.

  The experience indicates that a question based curriculum of guided discovery fits well with primary sources. Historical sources were usually written for experts of the period, not as teaching materials, and naturally evoke a wealth of questions. What could be more perfect? My challenge was to provide the right tasks and guidance.

- *Just-in-time (or just-as-needed):*

  I wanted motivation for new understanding always to come from investigation of Germain's writings. My goal was that students should always learn new things just-in-time, or even more strongly, just-as-needed. The drive for exploring something new should come from keeping one's eye firmly on the ball, namely Germain's research program. Only when students were stuck on something from Germain would I guide them to some new result that they needed to learn. My only exception would be sometimes to have students generalize something that had been discovered first for understanding Germain.

  I found that all desired course topics arose just-as-needed in understanding the manuscripts, though perhaps in different order than in a modern textbook. I forced myself to adhere to the just-as-needed maxim, and the result was highly satisfying. Students were always motivated by what was needed at each moment to advance with Germain.

  Primary sources are good for a just-as-needed pedagogy, since the barriers requiring new knowledge rise up naturally while studying a bigger picture.

- *No textbook in common:*

  I decided not to have a common textbook between me and my students, even as a supplement. There was a danger that I or my students would begin referring to the book in class interactions,

and the focus would then shift away from Germain. By not having a common book, I would force myself to prepare assignments with all eyes always on Germain's manuscripts, and thereby reinforce both the guided discovery and just-as-needed pedagogies.

However, I did not want my students to feel too anxious at a lack of resources, so I asked each student to choose a book of their own to have as a comforting security blanket, and said I would be happy to discuss their book material with them at any time. This seemed to work very well.

I found that having no single textbook in common fit extremely well with primary sources, since it kept the focus on the primary sources, while providing a comforting sense of security for students.

- *Mystery detectives:*

  I designed the course with student as detective, learning mathematics as needed to follow Germain's trail to prove Fermat's Last Theorem. The primary sources were well suited to this, partly because they were research manuscripts, so one could see problems being solved firsthand, and one needed to learn all the background to keep up with the trail. I believe many courses could be designed around primary sources as a detective mystery for solving big, interesting questions.

- *Reading in advance, preparatory work for an active classroom, then work to complete at home:*

  I have a three part non-lecture pedagogy to obviate lecture, described in detail at [11]. Students have three types of work to prepare for each class day, staggered over three consecutive units of material. First (Part A), they read new material for two class days hence, write questions about it or respond to my questions, and I receive these and read them, to help me prepare for that future class; my direction of in-class activity will be guided by these reading responses. Second (Part B), students prepare a mathematical assignment of medium level "warm-up" exercises, on material already previously read, to bring to class. Class time is spent first discussing the responses I received earlier to the reading, and then mostly on the warm-up work they have prepared for that same unit. It is discussed and dissected in groups, and as a whole class, presented on the board, etc. Third (Part C), an assignment of higher level "final" exercises is completed at home on material already worked through in class, for careful marking by me, and indicates the level of understanding reached by the student. New Parts A, B, C then continue for the next class day, each on different units of material.

  I found that this worked every bit as well in this course, with the Germain sources, as in my other courses. It fit particularly well with wanting to put students in the driver's seat to decipher Germain.

Here is an example of homework assigned on a single day early in the semester. It includes reading and writing in advance (6A) on new material for two class days hence (unit 6), warm-up exercises (5B) for the next class day (unit 5 in-class work), and final work (4C) to complete at home after today's classroom activities (unit 4) focused on the previously assigned Part 4B.

Homework 6A/5B/4C

Please write these three assignments on separate sheets

6A: Hand in at beginning of class next time.

Read and write your questions on the next paragraph(s) of Germain's manuscript A: "If for proving that ..." through "even more satisfying results".

5B: Prepare these to discuss together and present in class next time.

1. Look up the classification of Pythagorean triples, i.e., natural number solutions to the equation $x^2 + y^2 = z^2$. Their complete classification has been known for a long time. Write down the details of classifying primitive Pythagorean triples, i.e., those where x, y, z, have no common divisor.

4C: Hand in at beginning of class next time.

1. State and prove a theorem to justify Germain's claim that

Now since nothing prevents the successive assignment of an infinity of values to N, one can conclude from what precedes that there must exist an infinity of values of p for which the equation $x^p + y^p = z^p$ is impossible.

In other words, state a hypothesis making explicit what she claims in the preceding, and show how it would prove Fermat's Last Theorem for infinitely many values of p, by showing how it would lead to her italicized theorem at the beginning of Manuscript A.

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4   Conclusion

I found that guided discovery to learn mathematics "as needed" interacted extremely well with studying original sources. What were the particular challenges for me?

First, I had to prepare just the right assignments, and guidance for students, and be flexible based on what happened in the classroom. From non-lecture teaching I was already used to adjusting as I go along, and I knew Germain's manuscripts well, so I was mathematically equipped for this pedagogical task. The Germain manuscripts were conducive materials, always providing questions to further challenge my students.

Second, I had to resist the temptation to introduce new phenomena before they actually arose in Germain's writing. I found that as the topics arose naturally in Germain's manuscripts, I gained the resolve to let "just-as-needed" perform its function. Students were constantly motivated by each new challenge towards the big goal.

Teaching number theory to Sophie Germain's manuscripts was the most exciting teaching experience I have ever had, and the students rose to the challenge, embracing the experience from the beginning. We also read and discussed the book *Sophie's Diary* [8, 9], a fictional diary by Sophie Germain during the ages 13–17, teaching herself the mathematics she will need to gain the serious attention of

Lagrange at age 18 (as she did in real life), battling societal and familial pressure not to study mathematics because of her sex, and living in the middle of the French revolution outside her door in the heart of Paris.

I am writing a book for the course based on Germain's manuscripts, with tasks and guidance for student and instructor to follow Germain's path. I endeavor to leave almost all the proofs to the student, with ample guidance and optional further exercises.

Figure 5. Number theory cookies

My students should have the last word. But first, I explain the final exam period, where groups presented their work on proof and applications of the Quadratic Reciprocity Law. Students brought homebaked number theory cookies (Figures 5, 6). Try deciphering the icing on the cookies, which include ingredients of Germain's plan for proving Fermat's Last Theorem, quadratic reciprocity, Louis XVI at the guillotine, and an escargot.

Here are selected student comments from anonymous course evaluations:

"I have seen so many connections this semester between what we do in this course and the other math classes I've been taking, more than I have in any other course. I will definitely continue to study number theory on my own because of this class!"; "much more challenging than I expected"; "I will hold on to these papers forever"; "I loved this class"; "It was really cool to read and learn directly from primary sources like Germain's manuscripts and letters!"; "I really liked the high level of student participation"; "Dr. Pengelley's way of teaching will influence my way of teaching in future. What I learn is to be a good teacher, teacher need to work hard more than anyone else. To make student active learner, the role of teacher is very important."; "I truly enjoyed learning number theory in a historical context through Sophie Germain's manuscripts, while also learning the material of a normal number theory course. I also really liked reading 'Sophie's Diary' and thought it promoted some good discussions"; "the fact that every class was conducted in a way that invited open discussion meant that I was comfortable adding my thoughts or asking questions"; "I really like the way historical sources

More number theory cookies

were incorporated into the course, and that Dr. Pengelley has personally translated manuscripts from French and used them to help us learn number theory".

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